

Cohomological classification of Ann-categories

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Received April 7, 2011; accepted January 23, 2013

Abstract. An Ann-category is a categorification of rings. Regular Ann-categories were classified by Shukla cohomology of algebras. In this paper, we state and prove the precise theorem on classification of Ann-categories in the general case based on Mac Lane cohomology of rings.

AMS subject classifications: 18D10, 16E40

Key words: Ann-category, Ann-functor, categorical ring, Mac Lane cohomology, Shukla cohomology

1. Introduction

Categories with monoidal structures \oplus, \otimes (or *categories with distributivity constraints*) were originally considered by Laplaza in [4]. Kapranov and Voevodsky [3] omitted conditions related to the commutativity constraint with respect to \otimes in the axioms of Laplaza and called these categories *ring categories*.

In an alternative approach, monoidal categories can be “refined” to become *categories with group structure* if the objects are all invertible (see [5, 12]). When the underlying category is a *groupoid* (that is, every morphism is an isomorphism), we obtain the notion of *monoidal category group-like* [1], or *Gr-category* [14]. These categories can be classified by the cohomology group $H^3(\Pi, A)$ of groups.

In 1987, Quang [8] introduced the notion of *Ann-category* which is a categorification of rings. Ann-categories are symmetric Gr-categories (or Picard categories) equipped with a monoidal structure \otimes . Since all objects are invertible and all morphisms are isomorphisms, the axioms of an Ann-category are much fewer than those of a ring category (see [7]). The first two invariants of an Ann-category \mathcal{A} are the ring $R = \pi_0 \mathcal{A}$ of isomorphism classes of the objects in \mathcal{A} and the R -bimodule $M = \pi_1 \mathcal{A} = \text{Aut}_{\mathcal{A}}(0)$. Via the structure transport, we can construct an Ann-category of type (R, M) which is Ann-equivalent to \mathcal{A} . A family of constraints of \mathcal{A} induces a 5-tuple of functions $(\xi, \alpha, \lambda, \rho : R^3 \rightarrow M, \eta : R^2 \rightarrow M)$ satisfying certain relations. This 5-tuple is called a *structure* of an Ann-category of type (R, M) . Our purpose is to classify these categories by an appropriate cohomology group.

First we deal with the category of *regular* Ann-categories (they satisfy $c_{A,A} = id$ for all objects A) which arises from the ring extension problem. In [9], these categories were classified by the cohomology group H_{Sh}^3 of the ring R (regarded as an

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\mathbb{Z} -algebra) in the sense of Shukla [13] (that we misleadingly call Mac Lane-Shukla). This result shows the relation between the notion of a regular Ann-category and the theory of Shukla cohomology. Note that the structure $(\xi, \eta, \alpha, \lambda, \rho)$ of a regular Ann-category has an extra condition $\eta(x, x) = 0$ for the symmetry constraint. This condition is similar to the requirement $f \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} = 0$ so that a 3-cocycle f of Mac Lane cohomology has a realization [6].

In 2007, Jibladze and Pirashvili [2] introduced the notion of *categorical ring* as a slightly modified version of the notion of Ann-category and classified categorical rings by the cohomology group $H_{MaL}^3(R, M)$. The condition $(Ann - 1)$ and the compatibility of \otimes with associativity and commutativity constraints with respect to \oplus are replaced by the compatibility of \otimes with the “associativity - commutativity” constraint. We prove in [10] that the category of all Ann-categories is a subcategory of the category of all categorical rings. We also show that there exists a serious gap in the proof of Proposition 2.3 [2]. The authors of [2] did not prove the existence of the isomorphisms

$$A \otimes 0 \rightarrow 0, \quad 0 \otimes A \rightarrow 0,$$

so that the distributivity constraints induce the \otimes -functors which are compatible with the unit constraints. Thus the $\pi_0 \mathcal{A}$ -bimodule structure of the abelian group $\pi_1 \mathcal{A}$ cannot be deduced from axioms of a categorical ring, and therefore results on cohomological classification of categorical rings can not be stated precisely. In the appendix, we give an example of a categorical ring which is not an Ann-category, and prove that the classification theorem in [2] is wrong.

The main result of this paper is the cohomological classification theorem for Ann-categories (Theorem 12) in the general case. It is not only a continuation of the results in [9] and in [11], but it also gives a new interpretation of low-dimensional Mac Lane cohomology groups.

After this introductory Section 1, Section 2 is devoted to recalling some well-known results: i) the construction of an Ann-category of type (R, M) which is the reduced Ann-category of an arbitrary one and the determination of a structure on such an Ann-category of type (R, M) ; ii) the Mac Lane cohomology and the obstruction theory of Ann-functors. In Section 3 we prove that there is a bijection

$$\text{Struct}[R, M] \leftrightarrow H_{MacL}^3(R, M)$$

between the set of cohomology classes of structures on (R, M) and the Mac Lane cohomology group of the ring R with coefficients in the R -bimodule M , and therefore we obtain the precise theorem on classification of Ann-categories and Ann-functors.

In short, sometimes we write AB or $A.B$ instead of $A \otimes B$.

2. Ann-categories of type (R, M)

Let us recall some necessary concepts and facts in this section from [8, 9].

A monoidal category is called a *Gr-category* (or a *categorical group*) if every object is invertible and the background category is a groupoid. A *Picard category* (or a *symmetric categorical group*) is a Gr-category equipped with a symmetry constraint

which is compatible with associativity constraint.

2.1. Ann-categories and Ann-functors

Definition 1. *An Ann-category consists of*

- i) *a category \mathcal{A} together with two bifunctors $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$;*
- ii) *a fixed object $0 \in \mathcal{A}$ together with natural isomorphisms $\mathbf{a}_+, \mathbf{c}, \mathbf{g}, \mathbf{d}$ such that $(\mathcal{A}, \oplus, \mathbf{a}_+, \mathbf{c}, (0, \mathbf{g}, \mathbf{d}))$ is a Picard category;*
- iii) *a fixed object $1 \in \mathcal{A}$ together with natural isomorphisms $\mathbf{a}, \mathbf{l}, \mathbf{r}$ such that $(\mathcal{A}, \otimes, \mathbf{a}, (1, \mathbf{l}, \mathbf{r}))$ is a monoidal category;*
- iv) *natural isomorphisms $\mathfrak{L}, \mathfrak{R}$ given by*

$$\begin{aligned} \mathfrak{L}_{A,X,Y} &: A \otimes (X \oplus Y) \longrightarrow (A \otimes X) \oplus (A \otimes Y), \\ \mathfrak{R}_{X,Y,A} &: (X \oplus Y) \otimes A \longrightarrow (X \otimes A) \oplus (Y \otimes A) \end{aligned}$$

such that the following conditions hold:

(Ann – 1) *for $A \in \mathcal{A}$, the pairs $(L^A, \check{L}^A), (R^A, \check{R}^A)$ defined by*

$$\begin{aligned} L^A &= A \otimes - & R^A &= - \otimes A \\ \check{L}_{X,Y}^A &= \mathfrak{L}_{A,X,Y} & \check{R}_{X,Y}^A &= \mathfrak{R}_{X,Y,A} \end{aligned}$$

are \oplus -functors which are compatible with \mathbf{a}_+ and \mathbf{c} ;

(Ann – 2) *for all $A, B, X, Y \in \mathcal{A}$, the following diagrams commute*

$$\begin{array}{ccccc} (AB)(X \oplus Y) & \xleftarrow{\mathbf{a}_{A,B,X \oplus Y}} & A(B(X \oplus Y)) & \xrightarrow{id_A \otimes \check{L}^B} & A(BX \oplus BY) \\ \downarrow \check{L}^{AB} & & & & \downarrow \check{L}^A \\ (AB)X \oplus (AB)Y & \xleftarrow{\mathbf{a}_{A,B,X} \oplus \mathbf{a}_{A,B,Y}} & A(BX) \oplus A(BY) & & \\ \\ (X \oplus Y)(BA) & \xrightarrow{\mathbf{a}_{X \oplus Y,B,A}} & ((X \oplus Y)B)A & \xrightarrow{\check{R}^B \otimes id_A} & (XB \oplus YB)A \\ \downarrow \check{R}^{BA} & & & & \downarrow \check{R}^A \\ X(BA) \oplus Y(BA) & \xrightarrow{\mathbf{a}_{X,B,A} \oplus \mathbf{a}_{Y,B,A}} & (XB)A \oplus (YB)A & & \\ \\ (A(X \oplus Y)B) & \xleftarrow{\mathbf{a}_{A,X \oplus Y,B}} & A((X \oplus Y)B) & \xrightarrow{id_A \otimes \check{R}^B} & A(XB \oplus YB) \\ \downarrow \check{L}^A \otimes id_B & & & & \downarrow \check{L}^A \\ (AX \oplus AY)B & \xrightarrow{\check{R}^B} & (AX)B \oplus (AY)B & \xleftarrow{\mathbf{a} \oplus \mathbf{a}} & A(XB) \oplus A(YB) \end{array}$$

$$\begin{array}{ccc}
(A \oplus B)X \oplus (A \oplus B)Y & \xleftarrow{\check{L}^{A \oplus B}} (A \oplus B)(X \oplus Y) & \xrightarrow{\check{R}^{X \oplus Y}} A(X \oplus Y) \oplus B(X \oplus Y) \\
\downarrow \check{R}^X \oplus \check{R}^Y & & \downarrow \check{L}^A \oplus \check{L}^B \\
(AX \oplus BX) \oplus (AY \oplus BY) & \xrightarrow{\mathbf{v}} & (AX \oplus AY) \oplus (BX \oplus BY)
\end{array}$$

where $\mathbf{v} = \mathbf{v}_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \longrightarrow (U \oplus Z) \oplus (V \oplus T)$ is a unique morphism constructed from $\oplus, \mathbf{a}_+, \mathbf{c}, \text{id}$ of the symmetric monoidal category (\mathcal{A}, \oplus) ; (Ann – 3) for the unit $1 \in \mathcal{A}$ of the operation \otimes , the following diagrams commute

$$\begin{array}{ccc}
1(X \oplus Y) & \xrightarrow{\check{L}^1} & 1X \oplus 1Y \\
\searrow \downarrow 1_X \oplus Y & & \swarrow \downarrow 1_X \oplus 1_Y \\
& X \oplus Y &
\end{array}
\quad
\begin{array}{ccc}
(X \oplus Y)1 & \xrightarrow{\check{R}^1} & X1 \oplus Y1 \\
\searrow \downarrow r_X \oplus Y & & \swarrow \downarrow r_X \oplus r_Y \\
& X \oplus Y &
\end{array}$$

Since each of pairs (L^A, \hat{L}^A) , (R^A, \hat{R}^A) is an \oplus -functor which is compatible with the associativity constraint in the Picard category \mathcal{A} , it is also compatible with the unit constraint $(0, \mathbf{g}, \mathbf{d})$, so we obtain the following result.

Lemma 1. *In an Ann-category \mathcal{A} there exist unique isomorphisms*

$$\hat{L}^A : A \otimes 0 \longrightarrow 0, \hat{R}^A : 0 \otimes A \longrightarrow 0$$

such that the following diagrams commute

$$\begin{array}{ccc}
AX & \xleftarrow{L^A(\mathbf{g})} & A(0 \oplus X) \\
\uparrow \mathbf{g} & & \downarrow \check{L}^A \\
0 \oplus AX & \xleftarrow{\check{L}^A \oplus \text{id}} & A0 \oplus AX
\end{array}
\quad
\begin{array}{ccc}
AX & \xleftarrow{L^A(\mathbf{d})} & A(X \oplus 0) \\
\uparrow \mathbf{d} & & \downarrow \check{L}^A \\
AX \oplus 0 & \xleftarrow{\text{id} \oplus \check{L}^A} & AX \oplus A0
\end{array}$$

$$\begin{array}{ccc}
XA & \xleftarrow{R^A(\mathbf{g})} & (0 \oplus X)A \\
\uparrow \mathbf{g} & & \downarrow \check{R}^A \\
0 \oplus XA & \xleftarrow{\check{R}^A \oplus \text{id}} & 0A \oplus XA
\end{array}
\quad
\begin{array}{ccc}
XA & \xleftarrow{R^A(\mathbf{d})} & (X \oplus 0)A \\
\uparrow \mathbf{d} & & \downarrow \check{R}^A \\
XA \oplus 0 & \xleftarrow{\text{id} \oplus \check{R}^A} & XA \oplus 0A.
\end{array}$$

It is easy to see that if $(F, \check{F}, \hat{F}) : (\mathcal{A}, \oplus) \rightarrow (\mathcal{A}', \oplus)$ is a monoidal functor between two Gr-categories, then the canonical isomorphism $\hat{F} : F0 \rightarrow 0'$ can be deduced from others. Thus, we state the following definition.

Definition 2. *Let \mathcal{A} and \mathcal{A}' be Ann-categories. An Ann-functor $(F, \check{F}, \hat{F}, F_*) : \mathcal{A} \rightarrow \mathcal{A}'$ consists of a functor $F : \mathcal{A} \rightarrow \mathcal{A}'$, natural isomorphisms*

$$\check{F}_{X,Y} : F(X \oplus Y) \rightarrow F(X) \oplus F(Y), \quad \hat{F}_{X,Y} : F(X \otimes Y) \rightarrow F(X) \otimes F(Y),$$

and an isomorphism $F_* : F(1) \rightarrow 1'$ such that (F, \check{F}) is a symmetric monoidal functor with respect to the operation \oplus , (F, \hat{F}, F_*) is a monoidal functor with respect

to the operation \otimes , and $(F, \check{F}, \tilde{F})$ satisfies two following commutative diagrams

$$\begin{array}{ccccc}
 F(X(Y \oplus Z)) & \xrightarrow{\tilde{F}} & FX.F(Y \oplus Z) & \xrightarrow{id \otimes \tilde{F}} & FX(FY \oplus FZ) \\
 \downarrow F(\mathfrak{L}) & & & & \downarrow \mathfrak{L}' \\
 F(XY \oplus XZ) & \xrightarrow{\tilde{F}} & F(XY) \oplus F(XZ) & \xrightarrow{\tilde{F} \oplus \tilde{F}} & FX.FY \oplus FX.FZ \\
 \\
 F((X \oplus Y)Z) & \xrightarrow{\tilde{F}} & F(X \oplus Y).FZ & \xrightarrow{\tilde{F} \otimes id} & (FX \oplus FY)FZ \\
 \downarrow F(\mathfrak{R}) & & & & \downarrow \mathfrak{R}' \\
 F(XZ \oplus YZ) & \xrightarrow{\tilde{F}} & F(XZ) \oplus F(YZ) & \xrightarrow{\tilde{F} \oplus \tilde{F}} & FX.FZ \oplus FY.FZ.
 \end{array}$$

These diagrams are called the compatibility of the functor F with the distributivity constraints.

An Ann-morphism (or a homotopy)

$$\theta : (F, \check{F}, \tilde{F}, F_*) \rightarrow (F', \check{F}', \tilde{F}', F'_*)$$

between Ann-functors is an \oplus -morphism, as well as an \otimes -morphism.

If there exists an Ann-functor $(F', \check{F}', \tilde{F}', F'_*) : \mathcal{A}' \rightarrow \mathcal{A}$ and Ann-morphisms $F'F \xrightarrow{\sim} id_{\mathcal{A}}$, $FF' \xrightarrow{\sim} id_{\mathcal{A}'}$, we say that $(F, \check{F}, \tilde{F}, F_*)$ is an Ann-equivalence, and \mathcal{A} , \mathcal{A}' are Ann-equivalent.

It can be proved that each Ann-functor is an Ann-equivalence if and only if F is a categorical equivalence.

Lemma 2. Any Ann-functor $F = (F, \check{F}, \tilde{F}, F_*) : \mathcal{A} \rightarrow \mathcal{A}'$ is homotopic to an Ann-functor $F' = (F', \check{F}', \tilde{F}', F'_*)$, where $F'0 = 0'$, $\tilde{F}' = id_{0'}$, and $F'1 = 1'$, $F'_* = id_{1'}$.

Proof. Consider a family of isomorphisms in \mathcal{A}' :

$$\theta_X = \begin{cases} id_{FX} & \text{if } X \neq 0, X \neq 1, \\ \widehat{F} & \text{if } X = 0, \\ F_* & \text{if } X = 1, \end{cases}$$

for $X \in \mathcal{A}$. Then, the Ann-functor F' can be constructed in a unique way such that $\theta : F \rightarrow F'$ becomes a homotopy. Namely,

$$\begin{aligned}
F'X &= \begin{cases} FX & \text{if } X \neq 0, X \neq 1, \\ 0' & \text{if } X = 0, \\ 1' & \text{if } X = 1, \end{cases} \\
F'(f : X \rightarrow Y) &= \theta_Y F(f)(\theta_X)^{-1} : F'X \rightarrow F'Y, \\
\check{F}'_{X,Y} &= (\theta_X \oplus \theta_Y) \check{F}_{X,Y} \theta_{X \oplus Y}^{-1}, \\
\widehat{F}'_{X,Y} &= (\theta_X \otimes \theta_Y) \widehat{F}_{X,Y} \theta_{XY}^{-1}, \\
\widehat{F}' &= \widehat{F} \theta_0^{-1} = id_{0'}, \quad F'_* = F_* \theta_1^{-1} = id_{1'}.
\end{aligned}$$

□

Based on Lemma 2, we refer to $(F, \check{F}, \widehat{F})$ as an Ann-functor.

2.2. Reduced Ann-categories

For an Ann-category \mathcal{A} , the set $R = \pi_0 \mathcal{A}$ of isomorphism classes of the objects in \mathcal{A} is a ring where the operations $+$, \times are induced by \oplus , \otimes on \mathcal{A} , and $M = \pi_1 \mathcal{A} = \text{Aut}(0)$ is an abelian group where the operation, denoted by $+$, is just the composition. Moreover, $M = \pi_1 \mathcal{A}$ is an R -bimodule with the actions

$$sa = \lambda_X(a), \quad as = \rho_X(a),$$

where $X \in s, s \in \pi_0 \mathcal{A}, a \in \pi_1 \mathcal{A}$ and λ_X, ρ_X satisfy the commutative diagrams

$$\begin{array}{ccc}
X.0 & \xrightarrow{\hat{L}^X} & 0 \\
id \otimes a \downarrow & & \downarrow \lambda_X(a) \\
X.0 & \xrightarrow{\hat{L}^X} & 0,
\end{array}
\quad
\begin{array}{ccc}
0.X & \xrightarrow{\hat{R}^X} & 0 \\
a \otimes id \downarrow & & \downarrow \rho_X(a) \\
0.X & \xrightarrow{\hat{R}^X} & 0.
\end{array}$$

We recall briefly some main facts of the construction of the reduced Ann-category $S_{\mathcal{A}}$ of \mathcal{A} via the structure transport (for details, see [9]). The objects of $S_{\mathcal{A}}$ are the elements of the ring $\pi_0 \mathcal{A}$. A morphism is an automorphism $(s, a) : s \rightarrow s$, $s \in \pi_0 \mathcal{A}, a \in \pi_1 \mathcal{A}$. The composition of morphisms is given by

$$(s, a) \circ (s, b) = (s, a + b).$$

For each $s \in \pi_0 \mathcal{A}$, choose an object $X_s \in \mathcal{A}$ such that $X_0 = 0, X_1 = 1$, and choose an isomorphism $i_X : X \rightarrow X_s$ such that $i_{X_s} = id_{X_s}$. We obtain two functors

$$\begin{cases} G : \mathcal{A} \rightarrow S_{\mathcal{A}} \\ G(X) = [X] = s \\ G(X \xrightarrow{f} Y) = (s, \gamma_{X_s}^{-1}(i_Y f i_X^{-1})), \end{cases}
\quad
\begin{cases} H : S_{\mathcal{A}} \rightarrow \mathcal{A} \\ H(s) = X_s \\ H(s, a) = \gamma_{X_s}(a), \end{cases} \quad (1)$$

where $X, Y \in s$ and $f : X \rightarrow Y$, and γ_X is a map defined by the following commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\gamma_X(a)} & X \\
\mathbf{g}_X \uparrow & & \uparrow \mathbf{g}_X \\
0 \oplus X & \xrightarrow{a \oplus id} & 0 \oplus X
\end{array}$$

Diagram 1

The operations on $S_{\mathcal{A}}$ are defined by

$$\begin{aligned}
s \oplus t &= G(H(s) \oplus H(t)) = s + t, \\
(s, a) \oplus (t, b) &= G(H(s, a) \oplus H(t, b)) = (s + t, a + b), \\
s \otimes t &= G(H(s) \otimes H(t)) = st, \\
(s, a) \otimes (t, b) &= G(H(s, a) \otimes H(t, b)) = (st, sb + at),
\end{aligned}$$

where $s, t \in \pi_0 \mathcal{A}$, $a, b \in \pi_1 \mathcal{A}$. Obviously, these operations do not depend on the choice of the set of representatives (X_s, i_X) .

The constraints in $S_{\mathcal{A}}$ are defined by those in \mathcal{A} by means of the notion of *stick*. A *stick* in \mathcal{A} is a set of representatives (X_s, i_X) such that

$$\begin{aligned}
i_{0 \oplus X_t} &= \mathbf{g}_{X_t}, & i_{X_s \oplus 0} &= \mathbf{d}_{X_s}, \\
i_{1 \otimes X_t} &= \mathbf{l}_{X_t}, & i_{X_s \otimes 1} &= \mathbf{r}_{X_s}, & i_{0 \otimes X_t} &= \widehat{R}^{X_t}, & i_{X_s \otimes 0} &= \widehat{L}^{X_s}.
\end{aligned}$$

The unit constraints for two operations \oplus, \otimes in $S_{\mathcal{A}}$ are $(0, id, id)$ and $(1, id, id)$, respectively. The functor H and isomorphisms

$$\check{H} = i_{X_s \oplus X_t}^{-1}, \quad \widetilde{H} = i_{X_s \otimes X_t}^{-1} \quad (2)$$

transport the constraints $\mathbf{a}_+, \mathbf{c}, \mathbf{a}, \mathfrak{L}, \mathfrak{R}$ of \mathcal{A} to those $\xi, \eta, \alpha, \lambda, \rho$ of $S_{\mathcal{A}}$. Then, the category

$$(S_{\mathcal{A}}, \xi, \eta, (0, id, id), \alpha, (1, id, id), \lambda, \rho)$$

is an Ann-category which is equivalent to \mathcal{A} by the Ann-equivalence $(H, \check{H}, \widetilde{H}) : S_{\mathcal{A}} \rightarrow \mathcal{A}$. Besides, the functor $G : \mathcal{A} \rightarrow S_{\mathcal{A}}$ together with isomorphisms

$$\check{G}_{X,Y} = G(i_X \oplus i_Y), \quad \widetilde{G}_{X,Y} = G(i_X \otimes i_Y) \quad (3)$$

is also an Ann-equivalence. We refer to $S_{\mathcal{A}}$ as an Ann-category of *type* (R, M) , called a *reduction* of \mathcal{A} . We also call $(H, \check{H}, \widetilde{H})$ and $(G, \check{G}, \widetilde{G})$ *canonical* Ann-equivalences, the family of constraints $h = (\xi, \eta, \alpha, \lambda, \rho)$ of $S_{\mathcal{A}}$ a *structure* of the Ann-category of type (R, M) , or simply a *structure on* (R, M) .

The following result follows from the axioms of an Ann-category.

Theorem 1 ([9, Theorem 3.1]). *In the reduced Ann-category $S_{\mathcal{A}}$ of an Ann-category \mathcal{A} , the structure $(\xi, \eta, \alpha, \lambda, \rho)$ consists of functions with values in $\pi_1 \mathcal{A}$ such that for any $x, y, z, t \in \pi_0 \mathcal{A}$, the following conditions hold:*

$$A_1. \xi(y, z, t) - \xi(x + y, z, t) + \xi(x, y + z, t) - \xi(x, y, z + t) + \xi(x, y, z) = 0,$$

$$A_2. \xi(x, y, z) - \xi(x, z, y) + \xi(z, x, y) + \eta(x + y, z) - \eta(x, z) - \eta(y, z) = 0,$$

$$A_3. \eta(x, y) + \eta(y, x) = 0,$$

$$A_4. x\eta(y, z) - \eta(xy, xz) = \lambda(x, y, z) - \lambda(x, z, y),$$

$$A_5. \eta(x, y)z - \eta(xz, yz) = \rho(x, y, z) - \rho(y, x, z),$$

$$A_6. x\xi(y, z, t) - \xi(xy, xz, xt) = \lambda(x, z, t) - \lambda(x, y + z, t) + \lambda(x, y, z + t) - \lambda(x, y, z),$$

$$A_7. \xi(x, y, z)t - \xi(xt, yt, zt) = \rho(y, z, t) - \rho(x + y, z, t) + \rho(x, y + z, t) - \rho(x, y, z),$$

$$A_8. \rho(x, y, z + t) - \rho(x, y, z) - \rho(x, y, t) + \lambda(x, z, t) + \lambda(y, z, t) - \lambda(x + y, z, t) \\ = \xi(xz + xt, yz, yt) + \xi(xz, xt, yz) - \eta(xt, yz) + \xi(xz + yz, xt, yt) - \xi(xz, yz, xt),$$

$$A_9. \alpha(x, y, z + t) - \alpha(x, y, z) - \alpha(x, y, t) = x\lambda(y, z, t) + \lambda(x, yz, yt) - \lambda(xy, z, t),$$

$$A_{10}. \alpha(x, y + z, t) - \alpha(x, y, t) - \alpha(x, z, t) = x\rho(y, z, t) - \rho(xy, xz, t) + \lambda(x, yt, zt) \\ - \lambda(x, y, z)t,$$

$$A_{11}. \alpha(x + y, z, t) - \alpha(x, y, t) - \alpha(y, z, t) = -\rho(x, y, z)t - \rho(xz, yz, t) + \rho(x, y, zt),$$

$$A_{12}. x\alpha(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, zt) + \alpha(x, y, z)t = 0.$$

Further, these functions satisfy normalization conditions:

$$\xi(0, y, z) = \xi(x, 0, z) = \xi(x, y, 0) = 0,$$

$$\alpha(1, y, z) = \alpha(x, 1, z) = \alpha(x, y, 1) = 0,$$

$$\alpha(0, y, z) = \alpha(x, 0, z) = \alpha(x, y, 0) = 0,$$

$$\lambda(1, y, z) = \lambda(0, y, z) = \lambda(x, 0, z) = \lambda(x, y, 0) = 0,$$

$$\rho(x, y, 1) = \rho(0, y, z) = \rho(x, 0, z) = \rho(x, y, 0) = 0.$$

The induced operations on $S_{\mathcal{A}}$ do not depend on the choice of sticks. We now investigate the effect of different choices of the stick (X_s, i_X) in the induced constraints on $S_{\mathcal{A}}$.

Proposition 1. *Let \mathcal{S} and \mathcal{S}' be reduced Ann-categories of \mathcal{A} corresponding to the sticks (X_s, i_X) and (X'_s, i'_X) , respectively. Then the structures $(\xi, \eta, \alpha, \lambda, \rho)$ of \mathcal{S} and $(\xi', \eta', \alpha', \lambda', \rho')$ of \mathcal{S}' satisfy the following relations:*

$$A_{13}. \xi(x, y, z) - \xi'(x, y, z) = \tau(y, z) - \tau(x + y, z) + \tau(x, y + z) - \tau(x, y),$$

$$A_{14}. \eta(x, y) - \eta'(y, x) = \tau(x, y) - \tau(y, x),$$

$$A_{15}. \alpha(x, y, z) - \alpha'(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z,$$

$$A_{16}. \lambda(x, y, z) - \lambda'(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\tau(y, z) - \tau(xy, xz),$$

$$A_{17}. \rho(x, y, z) - \rho'(x, y, z) = \nu(x + y, z) - \nu(x, z) - \nu(y, z) + \tau(x, y)z - \tau(xz, yz),$$

where $\tau, \nu : (\pi_0 \mathcal{A})^2 \rightarrow \pi_1 \mathcal{A}$ are the functions satisfying the normalization conditions $\tau(0, y) = \tau(x, 0) = 0$ and $\nu(0, y) = \nu(x, 0) = \nu(1, y) = \nu(x, 1) = 0$.

Two structures $(\xi, \eta, \alpha, \lambda, \rho)$ and $(\xi', \eta', \alpha', \lambda', \rho')$ of Ann-categories of type (R, M) are *cohomologous* if and only if they satisfy the relations $A_{13} - A_{17}$ in Proposition 1.

Note that two unit constraints of \oplus and \otimes in an Ann-category of type (R, M) are both strict. It is easy to prove the following lemma.

Lemma 3. *Two structures h and h' are cohomologous if and only if there exists an Ann-functor $(F, \tilde{F}, \tilde{F}) : (R, M, h) \rightarrow (R, M, h')$, where $F = id_{(R, M)}$.*

2.3. Mac Lane cohomology groups of rings and obstruction theory

Let R be a ring and M an R -bimodule. From the definition of Mac Lane cohomology of rings [6], we obtain the description of elements in the cohomology group $H_{MaL}^3(R, M)$.

The group $Z_{MaL}^3(R, M)$ of 3-cocycles of R with coefficients in M consists of the quadruples $(\sigma, \alpha, \lambda, \rho)$ of the maps:

$$\sigma : R^4 \rightarrow M; \quad \alpha, \lambda, \rho : R^3 \rightarrow M$$

satisfying the following conditions:

$$\begin{aligned} M_1. & \quad x\alpha(y, z, t) - \alpha(xy, z, t) + \alpha(x, yz, t) - \alpha(x, y, zt) + \alpha(x, y, z)t = 0, \\ M_2. & \quad -\alpha(x, z, t) - \alpha(y, z, t) + \alpha(x + y, z, t) + \rho(xz, yz, t) - \rho(x, y, zt) + \rho(x, y, z)t = 0, \\ M_3. & \quad -\alpha(x, y, t) - \alpha(x, z, t) + \alpha(x, y + z, t) + x\rho(y, z, t) - \rho(xy, xz, t) - \lambda(x, yt, zt) \\ & \quad + \lambda(x, y, z)t = 0, \\ M_4. & \quad \alpha(x, y, z) + \alpha(x, y, t) - \alpha(x, y, z + t) + x\lambda(y, z, t) - \lambda(xy, z, t) + \lambda(x, yz, yt) = 0, \\ M_5. & \quad -\lambda(x, z, t) - \lambda(y, z, t) + \lambda(x + y, z, t) + \rho(x, y, z) + \rho(x, y, t) - \rho(x, y, z + t) \\ & \quad + \sigma(xz, xt, yz, yt) = 0, \\ M_6. & \quad \lambda(r, x, y) + \lambda(r, z, t) - \lambda(r, x + z, y + t) - \lambda(r, x, z) - \lambda(r, y, t) + \lambda(r, x + y, z + t) \\ & \quad - r\sigma(x, y, z, t) + \sigma(rx, ry, rz, rt) = 0, \\ M_7. & \quad -\rho(x, y, r) - \rho(z, t, r) + \rho(x + z, y + t, r) + \rho(x, z, r) + \rho(y, t, r) - \rho(x + y, z + t, r) \\ & \quad - \sigma(xr, yr, zr, tr) + \sigma(x, y, z, t)r = 0, \\ M_8. & \quad -\sigma(r, s, u, v) - \sigma(x, y, z, t) + \sigma(r + x, s + y, u + z, v + t) + \sigma(r, s, x, y) + \sigma(u, v, z, t) \\ & \quad - \sigma(r + u, s + v, x + z, y + t) - \sigma(r, u, x, z) - \sigma(s, v, y, t) \\ & \quad + \sigma(r + s, u + v, x + y, z + t) = 0. \end{aligned}$$

These functions satisfy normalization conditions:

$$\begin{aligned} \alpha(0, y, z) &= \alpha(x, 0, z) = \alpha(x, y, 0) = 0, \\ \lambda(0, y, z) &= \lambda(x, 0, z) = \lambda(x, y, 0) = 0, \\ \rho(0, y, z) &= \rho(x, 0, z) = \rho(x, y, 0) = 0, \\ \sigma(r, s, 0, 0) &= \sigma(0, 0, u, v) = \sigma(r, 0, u, 0) = \sigma(0, s, 0, v) = \sigma(r, 0, 0, v) = 0. \end{aligned}$$

The 3-cocycle $h = (\sigma, \alpha, \lambda, \rho)$ belongs to the group $B_{MaL}^3(R, M)$ if and only if there exist the functions $\tau, \nu : R^2 \rightarrow M$ satisfying:

$$\begin{aligned} M_9. & \quad \sigma(x, y, z, t) = \tau(x, y) + \tau(z, t) - \tau(x + z, y + t) - \tau(x, z) - \tau(y, t) + \tau(x + y, z + t) \\ M_{10}. & \quad \alpha(x, y, z) = x\nu(y, z) - \nu(xy, z) + \nu(x, yz) - \nu(x, y)z, \\ M_{11}. & \quad \lambda(x, y, z) = \nu(x, y + z) - \nu(x, y) - \nu(x, z) + x\tau(y, z) - \tau(xy, xz), \end{aligned}$$

$$M_{12}. \rho(x, y, z) = \nu(x + y, z) - \nu(x, z) - \nu(y, z) + \tau(x, y)z - \tau(xz, yz),$$

where τ, ν satisfy the normalization conditions: $\tau(0, y) = \tau(x, 0) = 0$ and $\nu(0, y) = \nu(x, 0) = \nu(1, y) = \nu(x, 1) = 0$.

The group $Z_{MaL}^2(R, M)$ consists of 2-cochains $g = (\tau, \nu)$ of the ring R with coefficients in the R -bimodule M satisfying

$$\partial g = 0.$$

The subgroup $B_{MaL}^2(R, M) \subset Z_{MaL}^2(R, M)$ of 2-coboundaries consists of the pairs (τ, ν) such that there exist the maps $t : R \rightarrow M$ satisfying $(\tau, \nu) = \partial_{MaL} t$, that is,

$$M_{13}. \tau(x, y) = t(y) - t(x + y) + t(x),$$

$$M_{14}. \nu(x, y) = xt(y) - t(xy) + t(x)y,$$

where t satisfies the normalization condition, $t(0) = t(1) = 0$.

The group $Z_{MaL}^1(R, M)$ consists of 1-cochains t of the ring R with coefficients in the R -bimodule M satisfying

$$\partial t = 0.$$

The subgroup of 1-coboundaries, $B_{MaL}^1(R, M) \subset Z_{MaL}^1(R, M)$, consists of the functions t such that there exists $a \in R$ satisfying $t(x) = ax - xa$.

The quotient group

$$H_{MaL}^i(R, M) = Z_{MaL}^i(R, M) / B_{MaL}^i(R, M), \quad i = 1, 2, 3,$$

is called the i^{th} Mac Lane cohomology group of the ring R with coefficients in the R -bimodule M .

Let us now recall some results on Ann-functors from [11]. Each Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ induces one S_F between their reduced Ann-categories. Throughout this section, let \mathcal{S} and \mathcal{S}' be Ann-categories of types (R, M, h) and (R', M', h') , respectively.

A functor $F : \mathcal{S} \rightarrow \mathcal{S}'$ is called a functor of type (p, q) if

$$F(x) = p(x), \quad F(x, a) = (p(x), q(a)),$$

where $p : R \rightarrow R'$ is a ring homomorphism and $q : M \rightarrow M'$ is a group homomorphism such that

$$q(xa) = p(x)q(a), \quad x \in R, a \in M.$$

The group M' can be regarded as an R -module with the action $xa' = p(x)a'$, so q is an R -bimodule homomorphism. In this case, we say that (p, q) is a pair of homomorphisms and that the function

$$k = q_* h - p^* h' \tag{4}$$

is an obstruction of F , where p^*, q_* are canonical homomorphisms,

$$Z_{MacL}^3(R, M) \xrightarrow{q_*} Z_{MacL}^3(R, M') \xleftarrow{p^*} Z_{MacL}^3(R', M').$$

Proposition 2 ([11, Proposition 4.3]). *Every Ann-functor $F : \mathcal{S} \rightarrow \mathcal{S}'$ is a functor of type (p, q) .*

Keeping in mind that γ is the map defined by Diagram 1, we state the following proposition.

Proposition 3 ([11, Proposition 4.1]). *Let \mathcal{A} and \mathcal{A}' be Ann-categories. Then every Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ induces an Ann-functor $S_F : S_{\mathcal{A}} \rightarrow S_{\mathcal{A}'}$ of type (p, q) , where*

$$\begin{aligned} p &= F_0 : \pi_0 \mathcal{A} \rightarrow \pi_0 \mathcal{A}', [X] \mapsto [FX], \\ q &= F_1 : \pi_1 \mathcal{A} \rightarrow \pi_1 \mathcal{A}', u \mapsto \gamma_{F_0}^{-1}(Fu). \end{aligned}$$

Further,

- i) F is an equivalence if and only if F_0, F_1 are isomorphisms.
- ii) The Ann-functor S_F satisfies the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ H \uparrow & & \downarrow G' \\ S_{\mathcal{A}} & \xrightarrow{S_F} & S_{\mathcal{A}'} \end{array}$$

where H, G' are canonical Ann-equivalences defined by (1), (2), (3).

Since $\check{F}_{x,y} = (\bullet, \tau(x, y))$, and $\tilde{F}_{x,y} = (\bullet, \nu(x, y))$, we call $g_F = (\tau, \nu)$ a pair of functions *associated* to (\check{F}, \tilde{F}) , and hence an Ann-functor $F : \mathcal{S} \rightarrow \mathcal{S}'$ can be regarded as a triple (p, q, g_F) . It follows from the compatibility of F with the constraints that

$$q_* h - p^* h' = \partial(g_F), \quad (5)$$

Moreover, Ann-functors (F, g_F) and $(F', g_{F'})$ are homotopic if and only if $F' = F$, that is, they are of the same type (p, q) , and there is a function $t : R \rightarrow M'$ such that $g_{F'} = g_F + \partial t$.

We write

$$\text{Hom}_{(p,q)}^{\text{Ann}}[\mathcal{S}, \mathcal{S}']$$

for the set of homotopy classes of Ann-functors of type (p, q) from \mathcal{S} to \mathcal{S}' .

Theorem 2 ([11, Theorem 4.4, 4.5]). *The functor $F : \mathcal{S} \rightarrow \mathcal{S}'$ of type (p, q) is an Ann-functor if and only if the obstruction $[k]$ vanishes in $H_{\text{MacL}}^3(R, M')$. Then, there exists a bijection*

$$\text{Hom}_{(p,q)}^{\text{Ann}}[\mathcal{S}, \mathcal{S}'] \leftrightarrow H_{\text{MacL}}^2(R, M'). \quad (6)$$

3. Classification of Ann-categories

In order to prove the main result (Theorem 3) of the paper, we first prove that the set of cohomology classes of structures on (R, M) and the group $H_{\text{MacL}}^3(R, M)$ are

coincident.

Lemma 4. *Each structure of an Ann-category of type (R, M) induces a 3-cocycle in $Z_{MaL}^3(R, M)$.*

Proof. Let $h = (\xi, \eta, \alpha, \lambda, \rho)$ be a structure of an Ann-category \mathcal{S} of type (R, M) . We define a function $\sigma : R^4 \rightarrow M$ by

$$\sigma(x, y, z, t) = \xi(x + y, z, t) - \xi(x, y, z) + \eta(y, z) + \xi(x, z, y) - \xi(x + z, y, t) \quad (7)$$

This equation shows that σ is just the morphism

$$\mathbf{v} : (x + y) + (z + t) \rightarrow (x + z) + (y + t)$$

in an Ann-category of type (R, M) .

First, the normalized property of σ follows from the ones of ξ and η

$$\sigma(0, 0, z, t) = \sigma(x, y, 0, 0) = \sigma(0, y, 0, t) = \sigma(x, 0, z, 0) = \sigma(x, 0, 0, t) = 0.$$

We now show that the quadruple $\hat{h} = (\sigma, \alpha, \lambda, \rho)$ satisfies the relations $M_1 - M_8$, and \hat{h} is therefore a 3-cocycle. The relation M_1 is just the relation A_{12} . The relations M_2, M_3, M_4, M_5 are just A_{11}, A_{10}, A_9, A_8 , respectively.

According to the coherence theorem in an Ann-category of type (R, M) , the following Diagrams 2 and 3 commute

$$\begin{array}{ccc} r[(x + y) + (z + t)] & \xrightarrow{id \otimes \mathbf{v}} & r[(x + z) + (y + t)] \\ \downarrow \mathcal{L} & & \downarrow \mathcal{L} \\ r(x + y) + r(z + t) & & r(x + z) + r(y + t) \\ \downarrow \mathcal{L} \oplus \mathcal{L} & & \downarrow \mathcal{L} \oplus \mathcal{L} \\ (rx + ry) + (rz + rt) & \xrightarrow{\mathbf{v}} & (rx + rz) + (ry + rt) \end{array}$$

Diagram 2

$$\begin{array}{ccc} [(r + s) + (u + v)] + [(x + y) + (z + t)] & \xrightarrow{\mathbf{v}} & [(r + s) + (x + y)] + [(u + v) + (z + t)] \\ \downarrow \mathbf{v} + \mathbf{v} & & \downarrow \mathbf{v} + \mathbf{v} \\ [(r + u) + (s + v)] + [(x + z) + (y + t)] & & [(r + x) + (s + y)] + [(u + z) + (v + t)] \\ \downarrow \mathbf{v} & & \downarrow \mathbf{v} \\ [(r + u) + (x + z)] + [(s + v) + (y + t)] & \xrightarrow{\mathbf{v} + \mathbf{v}} & [(r + x) + (u + z)] + [(s + y) + (v + t)] \end{array}$$

Diagram 3

These commutative diagrams imply the relations M_6, M_8 . The relation M_7 follows from a commutative diagram which is analogous to Diagram 2, where r is

tensored on the right-hand side. \square

Lemma 5. *Each Mac Lane 3-cocycle $(\sigma, \alpha, \lambda, \rho)$ is induced by a structure $(\xi, \eta, \alpha, \lambda, \rho)$ of an Ann-category of type (R, M) .*

Proof. Let $(\sigma, \alpha, \lambda, \rho)$ be an element in $Z_{MaL}^3(R, M)$. Set

$$\xi(x, y, z) = -\sigma(x, y, 0, z), \eta(x, y) = \sigma(0, x, y, 0),$$

we obtain a 5-tuple of functions $h = (\xi, \eta, \alpha, \lambda, \rho)$. The normalized properties of ξ, η follow from that of σ .

We now show that h is a structure of an Ann-category of type (R, M) . First, the relations $A_{12} - A_9$ are just $M_1 - M_4$. The relation A_1 follows from M_8 when $u = 0 = x = y = z$. The relation A_3 follows from M_8 when $r = s = v = 0 = x = z = t$. The relations A_4 and A_5 follow from M_6 and M_7 , respectively, when $x = t = 0$. The relations A_6 and A_7 follow from M_6 and M_7 , respectively, when $z = 0$.

To prove the relation A_2 , take $s = u = 0 = x = z = t$ in M_8 we obtain

$$-\xi(r, y, v) + \xi(r, v, y) - \eta(v, y) + \sigma(r, v, y, 0) = 0 \quad (8)$$

Now, take $r = u = 0 = y = z = t$ in M_8 we obtain

$$-\xi(x, s, v) + \eta(s, x) - \eta(s + v, x) + \sigma(s, v, x, 0) = 0.$$

In other words,

$$-\xi(y, r, v) + \eta(r, y) - \eta(r + v, y) + \sigma(r, v, y, 0) = 0 \quad (9)$$

Subtracting (9) from (8), we obtain the relation A_2 .

Finally, to prove the relation A_8 , note that σ can be presented by ξ, η as in (7). Indeed, take $v = 0 = x = y = z$ in M_8 we obtain

$$\sigma(r, s, u, t) + \xi(r + u, s, t) - \xi(r + s, u, t) - \sigma(r, s, u, 0) = 0. \quad (10)$$

Now, take $v = s, y = u$ in (9) we obtain

$$\xi(r, u, s) - \xi(r, s, u) - \eta(s, u) + \sigma(r, s, u, 0) = 0. \quad (11)$$

Adding (10) to (11) and doing some appropriate calculations, we get (7).

Because of (7), M_8 becomes A_8 . This means the 5-tuple of functions $h = (\xi, \eta, \alpha, \lambda, \rho)$ is a structure of an Ann-category of type (R, M) . Further, this structure induces the 3-cocycle $\hat{h} = (\sigma, \alpha, \lambda, \rho)$. \square

Lemma 6. *The structures h and h' of the Ann-category of type (R, M) are cohomologous if and only if the corresponding 3-cocycles \hat{h}, \hat{h}' are cohomologous.*

Proof. By Lemma 5, the structures h and h' induce elements \hat{h} and \hat{h}' in $Z_{MaL}^3(R, M)$, respectively. By Lemma 3, the functions $\alpha - \alpha', \lambda - \lambda', \rho - \rho'$ satisfy the relations $M_{10} - M_{12}$, where $\check{F} = \tau, \tilde{F} = \nu$. Besides, the following diagram commutes because of the coherence of a symmetric monoidal functor.

Note that $F = id$ and $\check{F} = \tau$, so the above commutative diagram implies

$$\begin{aligned} \sigma(x, y, z, t) - \sigma'(x, y, z, t) &= \tau(x + y, z + t) + \tau(x, y) + \tau(z, t) - \tau(x + z, y + t) \\ &\quad - \tau(x, z) - \tau(y, t). \end{aligned}$$

That means $\sigma - \sigma'$ satisfies M_9 . Thus, \hat{h} and \hat{h}' belong to the same cohomology class of $H_{MacL}^3(R, M)$.

Now, assume that $\hat{h} - \hat{h}' \in B_{MacL}^3(R, M)$. Then $\alpha - \alpha', \lambda - \lambda', \rho - \rho'$ satisfy $M_{10} - M_{12}$ which are just the relations $A_{15} - A_{17}$. By (7), the definition of σ and the normalized property of ξ, η , we have

$$\begin{aligned} \xi(x, y, z) &= -\sigma(x, 0, y, z), \quad \xi'(x, y, z) = -\sigma'(x, 0, y, z), \\ \eta(x, y) &= \sigma(0, x, y, 0), \quad \eta'(x, y) = \sigma'(0, x, y, 0). \end{aligned}$$

Therefore, A_{13}, A_{14} are obtained from M_9 , and thus h, h' are cohomologous structures. \square

Let $\text{Struct}[R, M]$ denote the set of cohomology classes of structures on (R, M) . Then, Lemmas 4, 5 and 6 lead to the following result.

Proposition 4. *There exists a bijection*

$$\begin{aligned} \text{Struct}[R, M] &\rightarrow H_{MacL}^3(R, M) \\ [h = (\xi, \eta, \alpha, \lambda, \rho)] &\mapsto [\hat{h} = (\sigma, \alpha, \lambda, \rho)] \end{aligned}$$

By the above lemma, we regard each cohomology class $[h] = [(\xi, \eta, \alpha, \lambda, \rho)]$ as an element of the group $H_{MacL}^3(R, M)$.

Let **Ann** refer to the category whose objects are Ann-categories, and whose morphisms are their Ann-functors.

We determine the category $\mathbf{H}_{\mathbf{Ann}}^3$ whose objects are triples $(R, M, [h])$, where $[h] \in H_{MacL}^3(R, M)$. A morphism $(R, M, [h]) \rightarrow (R', M', [h'])$ in $\mathbf{H}_{\mathbf{Ann}}^3$ is a pair (p, q) such that there exists a function $g : R^2 \rightarrow M'$ so that $(p, q, g) : (R, M, h) \rightarrow (R', M', h')$ is an Ann-functor, that is, $[p^*h'] = [q_*h] \in H_{MacL}^3(R, M')$. The composition in $\mathbf{H}_{\mathbf{Ann}}^3$ is defined by

$$(p', q') \circ (p, q) = (p'p, q'q).$$

Note that, Ann-functors $F, F' : \mathcal{A} \rightarrow \mathcal{A}'$ are homotopic if and only if $F_i = F'_i, i = 0, 1$ and $[g_F] = [g_{F'}]$ in $H_{MacL}^2(R, M)$. Denote by

$$\text{Hom}_{(p,q)}^{\text{Ann}}[\mathcal{A}, \mathcal{A}']$$

the set of homotopy classes of Ann-functors from \mathcal{A} to \mathcal{A}' inducing the same pair (p, q) , we prove the following classification result.

Theorem 3 (Classification Theorem). *There is a functor*

$$\begin{aligned} d : \mathbf{Ann} &\rightarrow \mathbf{H}_{\mathbf{Ann}}^3 \\ \mathcal{A} &\mapsto (\pi_0 \mathcal{A}, \pi_1 \mathcal{A}, [h_{\mathcal{A}}]) \end{aligned}$$

which has the following properties:

- i) dF is an isomorphism if and only if F is an equivalence.
- ii) d is surjective on objects.
- iii) d is full, but not faithful. For $(p, q) : d\mathcal{A} \rightarrow d\mathcal{A}'$, there is a bijection

$$\bar{d} : \text{Hom}_{(p,q)}^{\text{Ann}}[\mathcal{A}, \mathcal{A}'] \rightarrow H_{MacL}^2(\pi_0 \mathcal{A}, \pi_1 \mathcal{A}'). \quad (12)$$

Proof. In the Ann-category \mathcal{A} , for each stick (X_s, i_X) one can construct a reduced Ann-category $(\pi_0 \mathcal{A}, \pi_1 \mathcal{A}, h)$. If the choice of the stick is modified, then the 3-cocycle h changes to a cohomologous 3-cocycle h' . Therefore, \mathcal{A} uniquely determines an element $[h] \in H^3(\pi_0 \mathcal{A}, \pi_1 \mathcal{A})$.

For Ann-functors

$$\mathcal{A} \xrightarrow{F} \mathcal{A}' \xrightarrow{F'} \mathcal{A}'',$$

it can be seen that $d(F' \circ F) = dF' \circ dF$, and $d(id_{\mathcal{A}}) = id_{d\mathcal{A}}$. Therefore, d is a functor.

- i) According to Proposition 3.
- ii) If $(R, M, [h])$ is an object of $\mathbf{H}_{\mathbf{Ann}}^3$, then $\mathcal{S} = (R, M, h)$ is an Ann-category of type (R, M) , and obviously $d\mathcal{S} = (R, M, [h])$.
- iii) If (p, q) is a morphism in $\text{Hom}_{\mathbf{H}_{\mathbf{Ann}}^3}(d\mathcal{A}, d\mathcal{A}')$, then there is a function $g = (\tau, \nu)$, $\tau, \nu : (\pi_0 \mathcal{A})^2 \rightarrow \pi_1 \mathcal{A}'$ satisfying relation (5), and therefore

$$K = (p, q, g) : (\pi_0 \mathcal{A}, \pi_1 \mathcal{A}, h_{\mathcal{A}}) \rightarrow (\pi_0 \mathcal{A}', \pi_1 \mathcal{A}', h_{\mathcal{A}'})$$

is an Ann-functor. Thus, the composition $F = H'KG : \mathcal{A} \rightarrow \mathcal{A}'$ is an Ann-functor and $dF = (p, q)$. This shows that d is full.

In order to obtain the bijection (12), we prove that the correspondence

$$\begin{aligned} \Omega : \text{Hom}_{(p,q)}^{\text{Ann}}[\mathcal{A}, \mathcal{A}'] &\rightarrow \text{Hom}_{(p,q)}^{\text{Ann}}[S_{\mathcal{A}}, S_{\mathcal{A}'}] \\ [F] &\mapsto [S_F] \end{aligned} \quad (13)$$

is a bijection.

Clearly, if $F, F' : \mathcal{A} \rightarrow \mathcal{A}'$ are homotopic, then induced Ann-functors $S_F, S_{F'}$ are homotopic. Conversely, if S_F and $S_{F'}$ are homotopic, then the compositions $E = H'(S_F)G$ and $E' = H'(S_{F'})G$ are homotopic. Ann-functors E and E' are homotopic to F and F' , respectively. So, F and F' are homotopic. This shows that Ω is an injection.

Now, if $K = (p, q, g) : S_{\mathcal{A}} \rightarrow S_{\mathcal{A}'}$ is an Ann-functor, then the composition

$$F = H'KG : \mathcal{A} \rightarrow \mathcal{A}'$$

is an Ann-functor with $S_F = K$, that is, Ω is surjective. Now, the bijection (12) is the composition of (13) and (6). \square

Based on Theorem 3, Ann-categories having the same first two invariants can be classified up to equivalence.

Let R be a ring with a unit, M an R -bimodule which is regarded as a ring with null-multiplication. We say that the Ann-category \mathcal{A} has a *pre-stick of type* (R, M) if there is a pair of ring isomorphisms $\epsilon = (p, q)$

$$p : R \rightarrow \pi_0 \mathcal{A}, \quad q : M \rightarrow \pi_1 \mathcal{A}$$

which are compatible with the module action,

$$q(su) = p(s)q(u),$$

where $s \in R, u \in M$. The pair (p, q) is called a *pre-stick of type* (R, M) to the Ann-category \mathcal{A} .

A *morphism* between two Ann-categories $\mathcal{A}, \mathcal{A}'$ having pre-sticks of type (R, M) (with their pre-sticks are $\epsilon = (p, q)$ and $\epsilon' = (p', q')$, respectively) is an Ann-functor $(F, \check{F}, \tilde{F}) : \mathcal{A} \rightarrow \mathcal{A}'$ such that the following diagrams commute

$$\begin{array}{ccc} \pi_0 \mathcal{A} & \xrightarrow{F_0} & \pi_0 \mathcal{A}' \\ & \swarrow p \quad \searrow p' & \\ & R, & \end{array} \qquad \begin{array}{ccc} \pi_1 \mathcal{A} & \xrightarrow{F_1} & \pi_1 \mathcal{A}' \\ & \swarrow q \quad \searrow q' & \\ & M, & \end{array}$$

where (F_0, F_1) is a pair of homomorphisms induced by $(F, \check{F}, \tilde{F})$.

Clearly, it follows from the definition of an Ann-functor that F_0, F_1 are isomorphisms, therefore F is an equivalence.

Denote by

$$\mathbf{Ann}[R, M]$$

the set of equivalence classes of Ann-categories whose pre-sticks are of type (R, M) . One can prove the following result based on Theorem 3.

Theorem 4. *There is a bijection*

$$\begin{aligned} \Gamma : \mathbf{Ann}[R, M] &\rightarrow H_{MacL}^3(R, M) \\ [\mathcal{A}] &\mapsto q_*^{-1} p^* [h_{\mathcal{A}}] \end{aligned}$$

Proof. By Theorem 3, each Ann-category \mathcal{A} determines a unique element $[h_{\mathcal{A}}] \in H_{MacL}^3(\pi_0 \mathcal{A}, \pi_1 \mathcal{A})$, and hence an element

$$\epsilon[h_{\mathcal{A}}] = q_*^{-1} p^* [h_{\mathcal{A}}] \in H_{MacL}^3(R, M).$$

Now if $F : \mathcal{A} \rightarrow \mathcal{A}'$ is a functor between Ann-categories whose pre-sticks are of type (p, q) , then the induced Ann-functor $S_F = (p, q, g_F)$ satisfies the relation (5), and

therefore

$$p^*[h_{\mathcal{A}'}] = q_*[h_{\mathcal{A}}].$$

One can check that

$$\epsilon'[h_{\mathcal{A}'}] = \epsilon[h_{\mathcal{A}}].$$

This means Γ is a map. Moreover, it is an injection. Indeed, if $\Gamma[\mathcal{A}] = \Gamma[\mathcal{A}']$, then

$$\epsilon(h_{\mathcal{A}}) - \epsilon'(h_{\mathcal{A}'}) = \partial g.$$

Thus, there exists an Ann-functor J of type (id, id) from $\mathcal{I} = (R, M, \epsilon(h_{\mathcal{A}}))$ to $\mathcal{I}' = (R, M, \epsilon'(h_{\mathcal{A}'}))$. The composition

$$\mathcal{A} \xrightarrow{G} S_{\mathcal{A}} \xrightarrow{\epsilon^{-1}} \mathcal{I} \xrightarrow{J} \mathcal{I}' \xrightarrow{\epsilon'} S_{\mathcal{A}'} \xrightarrow{H'} \mathcal{A}'$$

shows that $[\mathcal{A}] = [\mathcal{A}']$, and Γ is an injection. Obviously, Γ is surjective. \square

In [9], the author proved that each structure of a regular Ann-category of type (R, M) (that is, a structure satisfies the *regular* condition, $\eta(x, x) = 0$) is an element in the group $Z_{Sh}^3(R, M)$ of Shukla 3-cocycles. From Classification Theorem 4.4 [9] and Theorem 3, the following result is obtained.

Corollary 1. *There is an injection*

$$H_{Sh}^3(R, M) \hookrightarrow H_{MacL}^3(R, M).$$

Appendix: A categorical ring which is not an Ann-category

Below, we construct a categorical ring which is not an Ann-category.

Let R be a ring with a unit and A an R -bimodule. Then, one constructs a categorical ring \mathcal{R} as follows. First, \mathcal{R} is a category defined as in Section 2. The objects of \mathcal{R} are elements of R , the morphisms in \mathcal{R} are automorphisms $(r, a) : r \rightarrow r$, $r \in R$, $a \in A$. Composition is the addition on A . Operations \oplus, \otimes on \mathcal{R} are given by

$$\begin{aligned} r \oplus s &= r + s, & (r, a) \oplus (s, b) &= (r + s, a + b), \\ r \otimes s &= rs, & (r, a) \otimes (s, b) &= (rs, rb + as). \end{aligned}$$

Suppose that the system $(\mathcal{R}, \oplus, \otimes)$ has a left distributivity constraint

$$\lambda_{r,s,t} : r(s + t) \rightarrow rs + rt$$

given by $\lambda_{r,s,t} = (\bullet, \lambda(r, s, t))$, where $\lambda : R^3 \rightarrow A$, and other constraints are strict. Then, the commutative diagrams in the axioms of a categorical ring are equivalent

to the equations

$$R_1. r\lambda(s, t, u) - \lambda(rs, t, u) + \lambda(r, st, su) = 0,$$

$$R_2. \lambda(r, s, t)u - \lambda(r, su, tu) = 0,$$

$$R_3. \lambda(1, s, t) = 0,$$

$$R_4. \lambda(r, s + t, u + v) + \lambda(r, s, t) + \lambda(r, u, v) = \lambda(r, s + u, t + v) + \lambda(r, s, u) + \lambda(r, t, v),$$

$$R_5. \lambda(r + r', s, t) = \lambda(r, s, t) + \lambda(r', s, t).$$

Let R be the ring of dual numbers on \mathbb{Z} , $R = \{a + b\epsilon \mid a, b \in \mathbb{Z}, \epsilon^2 = 0\}$ and $A = \mathbb{Z} \cong R/(\epsilon)$. Then, A is an R -bimodule with the natural actions

$$(a + b\epsilon)k = ak = k(a + b\epsilon).$$

The function $\lambda : R^3 \rightarrow A$, defined by

$$\lambda(a_r + b_r\epsilon, a_s + b_s\epsilon, a_t + b_t\epsilon) = b_r(a_s + a_t),$$

satisfies the equations $R_1 - R_5$, so that \mathcal{R} is a categorical ring.

It is clear that if $b_r \neq 0$ and $a_s \neq 0$, then $\lambda(r, 0, s) \neq 0$. Thus, by Theorem 1, \mathcal{R} is not an Ann-category.

One can deduce that:

1. Since the function λ is not normalized, $\hat{h} = (0, \lambda, 0, 0) \notin Z_{MacL}^3(R, A)$. This means that the classification theorem in [2] is wrong.
2. The condition (U) in the following theorem is necessary.

Theorem 5 (see [10]). *Each categorical ring \mathcal{R} is an Ann-category if and only if it satisfies the following condition;*

(U): *Each of pairs (L^A, \hat{L}^A) , (R^A, \hat{R}^A) , $A \in \mathcal{R}$, is an \oplus -functor which is compatible with the unit constraint $(0, \mathbf{g}, \mathbf{d})$ with respect to the operation \oplus .*

Acknowledgement

The author is much indebted to the referee, whose useful observations greatly improved my exposition.

References

- [1] A. FRÖHLICH, C. T. C. WALL, *Graded monoidal categories*, Compositio Math. **28**(1974), 229–285.
- [2] M. JIBLADZE, T. PIRASHVILI, *Third Mac Lane cohomology via categorical rings*, J. Homotopy Relat. Struct. **2**(2007), 187–216.
- [3] M. M. KAPRANOV, V. A. VOEVODSKY, *2-Categories and Zamolodchikov Tetrahedra Equations*, Proc. Sympos. Pure Math. **56**(1994), 177–259.
- [4] M. L. LAPLAZA, *Coherence for distributivity*, Lecture Notes Math. **281**(1972), 29–65.
- [5] M. L. LAPLAZA, *Coherence for Categories with Group Structure: an alternative approach*, J. Algebra **84**(1983), 305–323.

- [6] S. MAC LANE, *Extensions and obstruction for rings*, Illinois J. Math. **2**(1958), 316–345.
- [7] C. T. K. PHUNG, N. T. QUANG, N. T. THUY, *Relation between Ann-categories and ring categories*, Comm. Korean Math. Soc. **25**(2010), 523–535.
- [8] N. T. QUANG, *Introduction to Ann-categories*, J. Math. Hanoi **4**(1987), 14–24.
- [9] N. T. QUANG, *Structure of Ann-categories and Mac Lane-Shukla cohomology*, East-West J. Math. **5**(2003), 51–66.
- [10] N. T. QUANG, D. D. HANH, N. T. THUY, *On the Axiomatics of Ann-categories*, JP J. Algebra Number Theory Appl. **11**(2008), 59–72.
- [11] N. T. QUANG, D. D. HANH, *Homological classification of Ann-functors*, East-West J. Math. **11**(2009), 195–210.
- [12] N. SAAVEDRA RIVANO, *Catégories Tannakiennes*, Lecture notes in mathematics, Springer-Verlag, Berlin, 1972.
- [13] U. SHUKLA, *Cohomologie des algebras associatives*, Ann. Sci. Ecole Norm. Sup. **78**(1961), 163–209.
- [14] H. X. SINH, *Gr-catégories*, Thèse de doctorat, Université Paris VII, 1975.